Tunneling resonances and entanglement dynamics of cold bosons in the double well

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We present exact expressions for the quantum sloshing of ultracold bosons in a tilted two-well potential. Tunneling is suppressed by a small potential difference between wells, or tilt. However, tunneling resonances occur for critical values of the tilt when the barrier is high. At resonance, tunneling times on the order of 10-100 ms are possible. Furthermore, tunneling resonances constitute a dynamic scheme for creating robust few-atom entangled states in the presence of many bosons.

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Bose-Einstein condensates (BECs) in optical lattices are an ideal medium for studying a vast range of quantum many-body phenomena, including macroscopic quantum tunneling. A two-well potential is a simple limiting case which nevertheless exhibits rich quantum behavior. Spatially separate BECs in a two-well potential have recently been created in experiments [1] and tunneling times on the order of 50 ms have been observed [2]. The lifetime of a typical experiment is 1-100 s [3]. To describe these systems, a two-mode Bose-Hubbard-like Hamiltonian has often been employed [4, 5, 6, 7, 8]. However, these Hamiltonians assume that the trapping potential is symmetric. We study the dynamics of a system of ultracold bosons in a *tilted* double-well, i.e., a two-well trap with a nonzero potential difference between wells. Experimentally, tilt appears both as a systematic error and deliberately in device applications [9, 10, 11, 12]. Tilted optical lattices are especially relevant to gravitometry [9], quantum computing [13], and atomtronics [12].

In this Letter, we use a two-mode Hamiltonian to investigate the *quantum sloshing* of many bosons in a tilted double-well. That is, we present the tunneling dynamics of a system in which all atoms are initially localized in one well. Past studies have focused on the suppression of tunneling due to environmental effects such as finite temperature or coupling to a reservoir [14, 15, 16]. We have shown that tilt constitutes an additional source of decoherence [17]. Moreover, the self-trapping of a BEC in a two-well potential [2] has been attributed to long tunneling times [8]. Tilt displays radically different behavior than other forms of decoherence. Namely, *tunneling resonances* occur for critical values of the tilt when the barrier is high [17]. At resonance, tunneling between wells is hundreds of orders of magnitude faster and less sensitive to deviations in the tilt than in the symmetric case, as we will show. Furthermore, we present a simple scheme for the creation of few-atom entangled states both for few-body [11] and many-body systems [2]. Whereas past proposals involve ramping the barrier height [6] or continuous variation of atom-atom interactions via Feshbach resonance [5], entangled states are realized periodically in our scheme when all parameters are fixed.

A lattice of tilted double-wells has recently been created with the goal of realizing the first two-qubit logic gates made from neutral atoms [11]. In these systems, tilt is applied dynamically and plays a vital role in the desired logical operations. The analogs of N-type and P-type materials can be achieved by raising and lowering individual sites in an optical lattice [12]. Transistor-like behavior of a BEC has been demonstrated in a three-well trap [18]. The splitting of a BEC in an asymmetric double-well constitutes a novel gravity sensor [10] where tilt is introduced by a gravitational field gradient. Our two-well system models a truncated lattice, thereby providing insight into the dynamics of such systems.

The two-mode Hamiltonian for N weakly interacting bosons in a two-well potential is

$$\hat{H} = -J\sum_{j\neq j'} \hat{b}_j^\dagger \hat{b}_{j'} + U\sum_j \hat{n}_j\,(\hat{n}_j - 1) + \Delta V \hat{n}_L, \quad (1)$$

where the subscript $j \in \{L, R\}$ is the well or site index, J is the hopping strength, U is the interaction potential, and ΔV is the tilt. Here $\hat{b}_j$ and $\hat{b}_j^\dagger$ satisfy the usual bosonic annihilation and creation commutation relations and $\hat{n}_j \equiv \hat{b}_j^\dagger \hat{b}_j$. Eq. (1) can be derived from first principles quantum field theory for weakly interacting bosons at zero temperature [19]. We have previously discussed under what precise conditions Eq. (1) is sufficient to describe the two-well system [17]. An arbitrary state vector in Fock space is given by $|\psi\rangle = \sum_{n_L=0}^{N} c_{n_L}|n_L,\ N-n_L\rangle$, where n_L is the number of particles in the left well. We require the total number of particles N to be constant. Under this restriction, the Hamiltonian reduces to an $(N+1)\times(N+1)$ tridiagonal matrix [6] which we solve exactly. We consider the dynamics of a system in which all particles initially occupy the right well, i.e., $|\psi(t=0)\rangle = |0,\ N\rangle$. In the Schrödinger picture, the time evolved ket is $|\psi(t)\rangle \equiv \exp(-i\hat{H}t/\hbar)|\psi\rangle$. The probability of finding n_L particles in the left well at some time $t > 0$ is $P_{n_L}(t) \equiv |\langle n_L,\ N-n_L|\psi(t)\rangle|^2$, the average occupation of the left well is $\overline{n}_L(t) \equiv \langle\psi(t)|\hat{n}_L|\psi(t)\rangle$, and the average variance is $\sigma^2_{n_L}(t) \equiv \langle\psi(t)|\hat{n}_L^2|\psi(t)\rangle - \overline{n}_L^2$.

We first consider the simple case of noninteracting particles, $U = 0$, in a symmetric potential, $\Delta V = 0$, to illustrate the problem. The probability of finding all particles in the right well, i.e., $n_L = 0$, is

$$P_0(t) = \cos^{2N}(Jt/\hbar). \quad (2)$$

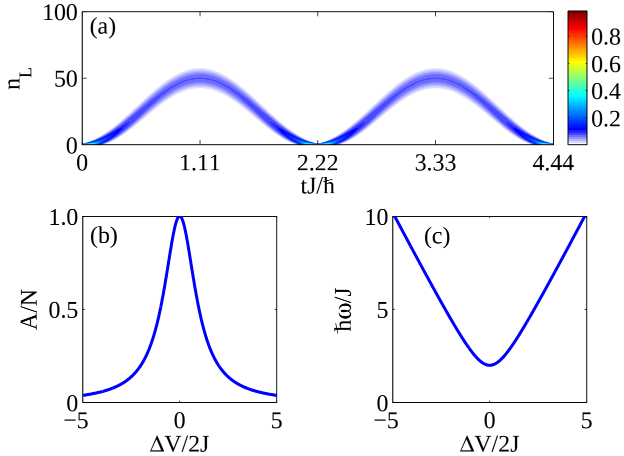

FIG. 1: *Suppression of tunneling for noninteracting atoms.* (a) Shown are the probability densities $P_{n_L}(t)$ for all number states when $N = 100$, $U = 0$, and $\Delta V = 2J$. Color indicates probability. Only $N/2 = 50$ particles tunnel between wells. (b) The tunneling amplitude and (c) the frequency of oscillation as a function of tilt. When $\Delta V > 2J\sqrt{N-1}$, tunneling is completely suppressed. Particles tunnel between wells faster in a tilted potential than in a symmetric potential.

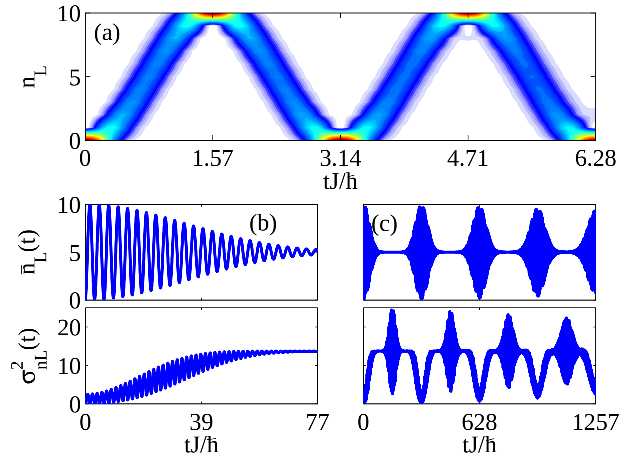

FIG. 2: *Damped tunneling in the low barrier regime.* (a) Shown are the probability densities $P_{n_L}(t)$ for all number states when $N = 10$ and $J/NU = 10$ for $t \ll T_{1/2}$. All particles tunnel between wells with period $T = \pi\hbar/J$. In (b) and (c), we show the average occupation (top panel) and the number variance (bottom panel) of the left well for longer times. (b) Oscillations between wells are damped by atom-atom interactions. (c) The first tunneling revival occurs when $t = T_r \equiv \pi\hbar/U$. The colorbar is the same as in Fig. 1.

The tunneling period is $T \equiv \pi\hbar/J$, which is independent of N. When $t = T/2$, the system is in state $|N, 0\rangle$ and all particles have tunneled into the left well. The average occupation and variance of the left well are

$$\overline{n}_L(t) = N\sin^2(Jt/\hbar), \qquad (3)$$
$$\sigma^2_{n_L}(t) = (N/4)\sin^2(2Jt/\hbar). \qquad (4)$$

The particles therefore tunnel sinusoidally between wells with a frequency $2J/\hbar$. The variance is greatest when $t = T/4$. At this time, the probability of finding n_L particles in the left well is

$$P_{n_L}(T/4) = 2^{-N}N!/[n_L!(N-n_L)!]. \qquad (5)$$

The system is in a *truncated coherent state*, i.e., a binomial superposition of all number-states.

When $\Delta V \neq 0$, the occupation of the left well is

$$\overline{n}_L(t) = A\sin^2(\omega t/2), \qquad (6)$$

where the amplitude and frequency of oscillation are

$$A \equiv N/[1 + (\Delta V/2J)^2], \qquad (7)$$
$$\omega \equiv (2J/\hbar)\sqrt{1 + (\Delta V/2J)^2}. \qquad (8)$$

When $\Delta V = 2J$, only $N/2$ particles tunnel between wells. Fig. 1(a) shows the probability densities $P_{n_L}(t)$ in this case. In Figs. 1(b) and 1(c), Eqs. (7) and (8) are plotted as a function of ΔV. Tunneling between wells is completely suppressed when $|\Delta V| > 2J\sqrt{N-1}$. Because the hopping strength J is much smaller than the barrier height [17], tunneling is highly sensitive to small tilt.

We now turn our attention to the interacting case when the barrier is low, $J \gg N|U|$, in a symmetric potential, $\Delta V = 0$. Whereas a single frequency $2J/\hbar$ characterizes $\overline{n}_L(t)$ when $U = 0$, N dominant frequencies emerge in this regime. The average occupation of the left well is given by the modulated signal

$$\overline{n}_L(t) = (N/2)\left[1 - \cos(2Jt/\hbar)\cos^{N-1}(Ut/\hbar)\right], \qquad (9)$$

to lowest order in NU/J. Here the high frequency carrier depends only on the hopping strength J while the low frequency envelope depends on both the interaction potential U and the total number of particles N. The envelope reaches half its maximum value when

$$t = T_{1/2} \equiv (\hbar/U)\cos^{-1}[2^{-1/(N-1)}]. \qquad (10)$$

At times $t \ll T_{1/2}$, all particles tunnel between wells with period T, as in Fig. 2(a). At times near $T_{1/2}$, on the other hand, only half the particles tunnel between wells with period T. When $t \simeq 2T_{1/2}$, there is essentially no tunneling (see Fig. 2(b)). Small interactions thus damp the oscillations between wells [8, 20]. However, tunneling revivals occur periodically with period $T_r \equiv \pi\hbar/U$. The first tunneling revival occurs when $|t - T_r| < T_{1/2}$, as shown in Fig. 2(c). The separation of time scales, $T_{1/2} \ll T_r$, occurs only for $N \gg 1$, as evident in Eq. (10).

For the remainder of our discussion, we turn to the high barrier limit, $J \ll |U|$, as it is key to the dynamic production of few-atom entangled states. We assume $U > 0$ without loss of generality with respect to the dynamics. In this regime, the two highest excited eigenstates are nearly-degenerate entangled cat states of the form $|\phi_{\pm}\rangle \equiv (|N,\,0\rangle \pm |0,\,N\rangle)/\sqrt{2}$ to lowest order in J/U [17]. Because the initial state $|\psi(0)\rangle = (|\phi_+\rangle - |\phi_-\rangle)/\sqrt{2}$ is a superposition of two eigenstates, the dynamics are described by the two-state system. The characteristic frequency is $\omega_N = \Delta E_N/\hbar$, where

$$\Delta E_N = 4U(J/2U)^N N/[(N-1)!], \tag{11}$$

is the energy difference of the states $|\phi_{\pm}\rangle$. As ΔE_N is a very small number, ω_N is also very small.

All particles occupy the right well with probability

$$P_0(t) = 1 - P_N(t) = \cos^2(\omega_N t/2), \tag{12}$$

In Fig. 3(a), we plot the probability densities $P_{n_L}(t)$ and the average occupation $\overline{n}_L(t)$ as a function of time. The tunneling period is $T_N \equiv 2\pi/\omega_N$. The average occupation and variance are

$$\overline{n}_L(t) = N\sin^2(\omega_N t/2), \tag{13}$$

$$\sigma^2_{n_L}(t) = (N^2/4)\sin^2(\omega_N t). \tag{14}$$

In this regime, as in the noninteracting case, all N particles oscillate sinusoidally between wells. There are two important differences. The first is that, when $J \ll U$, the period of oscillation depends on N and can become quite small for large values of N. Second, at time $t = T_N/4$, we find that $P_N = P_0 = 1/2$. At this time, all particles simultaneously occupy both wells and the system is described by an entangled Schrödinger cat state.

In order to characterize entanglement at $t = T_N/4$, we utilize three standard entanglement measures: Meyer's Q-measure [21, 22], the Shannon entropy S, which is equivalent to the entropy of entanglement [23] and the Schmidt rank k [24]. Meyer's Q-measure is given by $Q = [(N+1)/N][1 - 1/2(\mathrm{Tr}\rho_L^2 + \mathrm{Tr}\rho_R^2)]$, where $\rho_{L(R)} = \mathrm{Tr}_{R(L)}|\psi(t)\rangle\langle\psi(t)|$. The entropy is $S = -\sum_{n_L=0}^{N} P_{n_L}(t)\log_{N+1} P_{n_L}(t)$, and the Schmidt rank k is given by the number of non-zero eigenvalues of the reduced density matrix ρ_L. Here $Q = S = 0$ and $k = 1$ if and only if $|\psi(t)\rangle$ is a pure state. This occurs at $t = T_N/2$ and T_N. At time $T_N/4$, we find that each measure reaches a maximum value of $Q = N/[2(N+1)]$, $S = \log_{N+1}(2)$, and $k = 2$.

When the barrier is high, tunneling between wells is extremely sensitive to tilt ΔV. A small tilt causes the decoherence of the entangled eigenstates. When $\Delta V > 2\Delta E_N/N$, the highest excited eigenstates become number-states of the form $|0,\,N\rangle$ and $|N,\,0\rangle$ [17]. In this case, the initial condition $|\psi(0)\rangle$ is stationary and tunneling between wells is therefore suppressed. This is quite different from suppression of tunneling due to thermal effects or coupling to a reservoir [14, 15, 16]; our system is closed and suppression is due to an internal parameter, namely, imperfections in the trapping potential.

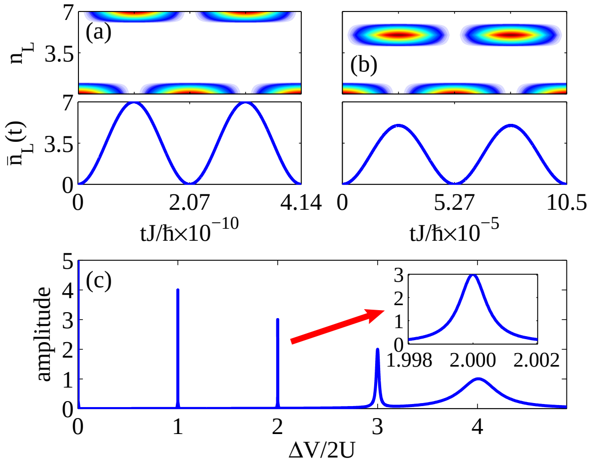

FIG. 3: *Tunneling resonances in a few-atom system.* Shown are the probability densities $P_{n_L}(t)$ when $N = 7$ and $J/U = 0.1$, for (a) $\Delta V = 0$ and (b) $\Delta V = 4U$. (a) All particles tunnel between wells with period T_N. At time $t = T_N/4$, the system is described by a cat state. (b) Only $N - 2 = 5$ particles tunnel between wells. The oscillation frequency is 5 orders of magnitude faster than the symmetric case. (c) Tunneling amplitude as a function of tilt ΔV for $N = 5$ and $J/U = 0.1$. Tunneling resonances occur when $\Delta V = \Delta V_p \equiv 2pU$. At resonance, $N - p$ particles tunnel between wells. The insert is a zoom around $\Delta V/2U = 2$.

Tunneling resonances occur when the tilt can be exactly compensated by atom-atom interactions. This happens when $\Delta V = \Delta V_p \equiv 2pU$ for $p \in \{1,\,2,\,\ldots,\,N-1\}$. In this case, the potential difference can be exactly compensated by the repulsive interaction of p particles in the lower well. Entangled eigenstates of the form $|\phi_{\pm};p\rangle \equiv (|N-p,\,p\rangle \pm |0,\,N\rangle)/\sqrt{2}$ reappear [17]. At resonance, the tunneling frequency is $\omega_N^p = \Delta E_N^p/\hbar$ where

$$\Delta E_N^p = \frac{4U(J/2U)^{N-p}(N-p)}{(N-p-1)!}\sqrt{\frac{N!}{p!(N-p)!}}, \tag{15}$$

is the level splitting between the states $|\phi_{\pm};p\rangle$. The average occupation of the left well is

$$\overline{n}_L(t) = (N-p)\sin^2(\omega_N^p t/2). \tag{16}$$

Here, $N - p$ particles tunnel between wells with period $T_N^p = 2\pi/\omega_N^p$. At time $t = T_N^p/2$, $N - p$ particles are in the left well. To compensate the tilt, p particles remain in the right well at all times. When $t = T_N^p/4$, the system is described by an entangled state such that $P_p = P_0 = 1/2$ and the entanglement measures Q, S, and k reach the same values as in the symmetric case. In Fig. 3(b) the tunneling dynamics for the second resonance, i.e., $p = 2$, are illustrated for a system of $N = 7$ particles. Near a resonance, tunneling is suppressed when

$$|\Delta V - \Delta V_p| > 2\Delta E_N^p/(N-p), \tag{17}$$

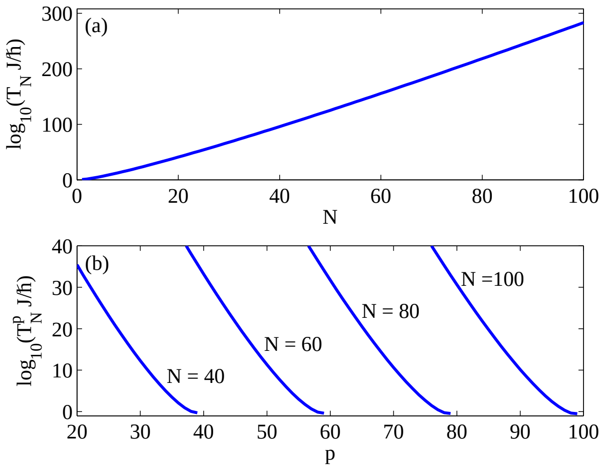

FIG. 4: *Tunneling periods in a many-body system.* (a) Shown is the tunneling period T_N versus the total number of particles N when $J/U = 0.1$ and $\Delta V = 0$. For large N, tunneling becomes very slow. (b) At resonance, $\Delta V = \Delta V_p \equiv 2pU$, only $N-p$ particles tunnel between wells. Shown are the tunneling periods T_N^p versus p for $N = 40$ to 100 with $J/U = 0.1$. At resonance, the oscillations can be hundreds of orders of magnitude faster than in the symmetric case.

as shown in Fig. 3(c). Note that tunneling resonances only occur for tilt applied to the left well.

Because ΔE_N^p is greater than ΔE_N by many orders of magnitude, tunneling near resonance is both much faster and less sensitive to tilt than tunneling in a symmetric potential. In Fig. 4(a), we show the symmetric tunneling period T_N versus N when $J/U = 0.1$. Clearly, T_N becomes very long as N becomes large. For instance, in a typical symmetric double-well used in experiments [11], 200 ^{87}Rb atoms tunnel between wells with period $T_{200} = 1.15 \times 10^{635}$ ms when $J/U = 0.0964$. Furthermore, tunneling is completely suppressed for deviations in the tilt greater than 4.16×10^{-636} nK $\cdot\, k_B$. Obviously, one does not expect to observe many-body tunneling in this regime. Under the same conditions, systems with as few as $N = 1$, 2, and 3 ^{87}Rb atoms yield tunneling times as long as $T_1 = 466$ ms, $T_2 = 4840$ ms, and $T_3 = 134000$ ms, respectively. Even in a few-particle system, tunneling times can be prohibitively long.

However, tunneling at resonance can be hundreds of orders of magnitude faster than the symmetric case, as in Fig. 4(b). For the 200-atom system discussed above, when $p = 197$, we find that $N - p = 3$ particles tunnel between wells with period $T_{200}^{197} = 117$ ms. This resonance occurs when $\Delta V = \Delta V_{197} = 210$ nK $\cdot\, k_B$. An entangled state of the form $|\psi\rangle = (|3, 197\rangle - i|0, 200\rangle)/\sqrt{2}$ will be realized at $T_{200}^{197}/4 = 29.25$ ms. Likewise, we find that $T_{200}^{198} = 34.3$ ms and $T_{200}^{199} = 33.0$ ms when $\Delta V = \Delta V_{198} = 211$ nK $\cdot\, k_B$ and $\Delta V_{199} = 212$ nK $\cdot\, k_B$, respectively. At resonance, this system is sensitive to deviations in the tilt on the order of 0.273 nK $\cdot\, k_B$, 1.40 nK $\cdot\, k_B$, and 2.90 nK$\cdot k_B$ for $p = 197$, 198, and 199, respectively. Thus, the observation of the tunneling of a few ^{87}Rb atoms is made possible by tunneling resonances in a many-body system.

In conclusion, we used the two-mode approximation to develop a Fock space picture of a system of ultracold bosons in a tilted two-well potential. A small tilt causes the complete suppression of tunneling. In the high barrier limit, long tunneling times prevent the observation of many-body tunneling even in a symmetric potential. However, in this regime, tunneling resonances occur when the tilt can be compensated by atom-atom interactions. At resonance, tunneling is much faster and less sensitive to tilt than in a symmetric potential. Furthermore, tunneling resonances can be used to create few-particle entangled states in a many-body system.

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